MULTIARC AND CURVE GRAPHS ARE HIERARCHICALLY HYPERBOLIC

MICHAEL C. KOPRESKI

Abstract. A multiarc and curve graph is a simplicial graph whose vertices are arc and curve systems on a compact, connected, orientable surface Σ. We show that all connected, non-trivial multiarc and curve graphs preserved by the natural action of $\mathrm{PMod}(\Sigma)$ and whose adjacent vertices have bounded geometric intersection number are hierarchically hyperbolic spaces with respect to witness subsurface projection. This result extends work of Kate Vokes on twist-free multicurve graphs and confirms two conjectures of Saul Schleimer in a broad setting. In addition, we prove that the $\mathrm{PMod}(\Sigma)$-equivariant quasi-isometry type of such a graph is uniquely classified by its set of connected witness subsurfaces.

1. Introduction and main results

Let Σ be a compact, connected, orientable, non-pants surface possibly with boundary such that $\chi(\Sigma) \leq -1$. We consider a broad class of natural combinatorial models built from collections of arc and curve systems on Σ:

Definition 1.1. A *multiarc and curve graph* $\mathcal{A}$ on Σ is a simplicial graph whose vertices are collections of disjoint isotopy classes of simple, essential arcs and curves in Σ. $\mathcal{A}$ is a *multicurve graph* if $V(\mathcal{A})$ consists of only multicurves.

Definition 1.2. A connected multiarc and curve graph $\mathcal{A}$ on Σ is *admissible* if

(i) $\mathrm{PMod}(\Sigma)$ preserves $V(\mathcal{A})$ and extends to an action on $\mathcal{A}$, and
(ii) if $(a, b) \in E(\mathcal{A})$, then the geometric intersection $i(a, b)$ is uniformly bounded.

Recall that a *witness subsurface* for $\mathcal{A}$ is an essential, non-pants subsurface that meets every vertex of $\mathcal{A}$. We prove the following:

Theorem 1.3. *Let $\mathcal{A}$ be an admissible multiarc and curve graph on Σ and let $\mathscr{X}$ denote its collection of witnesses. Then $(\mathcal{A}, \mathscr{X})$ is a hierarchically hyperbolic space with respect to subsurface projection to witness curve graphs $\mathcal{C}W$, $W \in \mathscr{X}$.*

Theorem 1.4. *The* $\mathrm{PMod}(\Sigma)$*-equivariant quasi-isometry type of an admissible multiarc and curve graph on Σ is uniquely determined by its connected witnesses.*

We first show the same for *partial marking graphs*, whose vertex set consists of markings in the sense of [MM00] which are "locally" complete and clean, in Section 2. Admissible partial marking graphs include the 1-skeleton $\mathcal{MC}^{(1)}(\Sigma)$ of Masur and Minsky's marking complex. Here, Theorem 1.4 defines a bijection:

E-mail address: `kopreski@math.utah.edu`.

any set of essential, non-pants subsurfaces closed under enlargement and the action of $\mathrm{PMod}(\Sigma)$ is the witness set for some admissible partial marking graph. In Section 3, we show that any admissible multiarc and curve graph on Σ is $\mathrm{PMod}(\Sigma)$-equivariantly quasi-isometric to an admissible partial marking graph on Σ, via a map coarsely preserving subsurface projection, whence Theorems 1.3 and 1.4 follow.

The present work is indebted to the excellent paper by Kate Vokes on the hierarchical hyperbolicity of *twist-free* multicurve graphs [Vok22], from which our results borrow both inspiration and overall strategy.

1.1. Background.

Given $\Sigma = \Sigma_g^b$ a compact, connected, orientable surface with genus g and b boundary components, let $\xi(\Sigma) = 3g + b - 3$ denote the *complexity* of Σ, equal to the number of components in a maximal multicurve. For a multiarc and curve graph $\mathcal{A}$ on a compact surface Σ, we define witness subsurfaces in the usual way:

Definition 1.5. A compact, essential (π_1-injective, non-peripheral) subsurface $W \subset \Sigma$ is a *witness* for $\mathcal{A}$ if $W \not\cong \Sigma_0^3$ and every arc and curve system in $V(\mathcal{A})$ intersects W.

Let $\pi_W : \mathcal{A} \to 2^{\mathcal{C}W} \setminus \{\varnothing\}$ denote subsurface projection and for $a, b \in V(\mathcal{A})$, let $d_W(a, b) := \mathrm{diam}_{\mathcal{C}W}(\pi_W(a) \cup \pi_W(b))$. Note that we do not require witnesses to be connected: if $W = V \sqcup V'$, let $\mathcal{C}W$ be the graph join $\mathcal{C}V * \mathcal{C}V'$.

We consider two conjectures of Saul Schleimer [Sch, §2.3] on the geometry of multiarc and curve graphs: the first characterizes δ-hyperbolicity, and the second proposes a distance formula in the sense of [MM00].

Conjecture 1.6 (Schleimer). *Let $M \geq 0$ and $\mathcal{A}$ be a multiarc and curve graph such that $(a, b) \in E(\mathcal{A})$ if and only if $i(a, b) \leq M$. $\mathcal{A}$ is δ-hyperbolic if and only if it does not admit disjoint connected witnesses.*

Conjecture 1.7 (Schleimer). *Let $M \geq 0$ and $\mathcal{A}$ be a multiarc and curve graph such that $(a, b) \in E(\mathcal{A})$ if and only if $i(a, b) \leq M$, and let $\mathscr{X}$ denote the collection of witnesses of $\mathcal{A}$. Then there exists $C' = C'(\mathcal{A})$ such that for any $C > C'$, there are constants $K, E \geq 0$ such that*

$$d_{\mathcal{A}}(a, b) \stackrel{K,E}{\asymp} \sum_{W \in \mathscr{X}} [d_W(a, b)]_C \ .$$

If a multiarc and curve graph $\mathcal{A}$ on Σ satisfies (i) of Definition 1.2, then we will say that $\mathcal{A}$ has a *natural* $\mathrm{PMod}(\Sigma)$ action. We note that Schleimer does not assume a natural action of $\mathrm{PMod}(\Sigma)$, and indeed there exist interesting complexes lacking such an action. Nonetheless, assuming a natural $\mathrm{PMod}(\Sigma)$ action and the connectivity of $\mathcal{A}$ (as we will for the remaining work), we observe that it is equivalent in the conjectures above to require only that $i(a, b)$ be uniformly bounded for $(a, b) \in E(\mathcal{A})$. In particular:

Remark 1.8. Let $\mathcal{A}$ be a connected multiarc and curve graph with a natural $\mathrm{PMod}(\Sigma)$ action, and suppose $\mathcal{A}'$ is obtained by adding edges between vertices a, b such that $i(a, b) < M$. Then $\mathcal{A} \underset{\text{q.i.}}{\cong} \mathcal{A}'$.

Rather than approaching Conjectures 1.6 and 1.7 directly, we appeal to a broader geometric property: if $\mathcal{A}$ is hierarchically hyperbolic with the usual witness subsurface projection structure, then both conjectures follow immediately. We will introduce hierarchical hyperbolicity and make precise these arguments at the end of the section.

1.1.1. *Low-complexity cases.* Note that $\xi(\Sigma) > 0$ by assumption. When $\xi(\Sigma) \leq 0$, any multiarc and curve graph on Σ is empty or it is finite and Σ admits no essential (non-peripheral) non-pants subsurfaces, except when $\Sigma \cong \Sigma_1^0$. In the latter case, any admissible multiarc and curve graph is quasi-isometric to the curve graph and Σ_1^0 is the only witness, hence Theorems 1.3 and 1.4 hold trivially.

1.1.2. *Twist-free multicurve graphs.* In the sense above, Conjectures 1.6 and 1.7 were resolved positively for a broad class of examples by Kate Vokes in [Vok22].

Definition 1.9 (Vokes)**.** A multicurve graph $\mathcal{G}$ on Σ is *twist-free* if it is admissible[1] and admits no annular witnesses.

Theorem 1.10 (Vokes)**.** *Let $\mathcal{G}$ be a twist-free multicurve graph. Then $(\mathcal{G}, \mathscr{X})$ is a hierarchically hyperbolic space with respect to subsurface projections to witness curve graphs $\mathcal{C}W$, $W \in \mathscr{X}$.*

Theorem 1.4 and in fact a bijection analogous to Theorem 2.12 below follow easily from Vokes' results in the special case of twist-free multicurve graphs.

The existence of annular witnesses appears related to non-hyperbolicity, or equivalently by Conjecture 1.6, the existence of disjoint connected witnesses. In particular, if $\xi(\Sigma) > 1$ and a multiarc and curve graph $\mathcal{A}$ on Σ admits an annular witness A, then $\mathcal{A}$ admits a disjoint pair of connected witnesses, namely A and a complementary component. As a partial converse, if $\mathcal{A}$ is an arc and curve graph that admits two disjoint witnesses W, W' separated by a unique complementary component, then $\mathcal{A}$ admits an annular witness. For example, given Γ a relation on $\pi_0(\partial\Sigma)$ the Γ-*prescribed arc graph* $\mathcal{A}(\Sigma, \Gamma)$, defined by the author in [Kop23], is the full subcomplex of $\mathcal{A}(\Sigma)$ spanned by arcs between boundary components in Γ. By passing to projections to $\mathcal{C}\Sigma$, $\mathcal{A}(\Sigma, \Gamma)$ may be viewed as an admissible multicurve graph; however, whenever Γ is bipartite (in particular, whenever $\mathcal{A}(\Sigma, \Gamma)$ is non-hyperbolic), it may be seen that $\mathcal{A}(\Sigma, \Gamma)$ admits $g-1$ pairwise disjoint annular witnesses where g is the genus of Σ, hence Theorem 1.10 does not apply.

Nonetheless, from Theorem 1.3 it follows immediately that $\mathcal{A}(\Sigma, \Gamma)$ is hierarchically hyperbolic with respect to the usual witness subsurface projection structure. As an immediately application, let Ω be an orientable surface of infinite topological type. We recall the *grand arc graph* $\mathcal{G}(\Omega)$ defined in [BNV22], whose vertices are arcs between ends of distinct maximal type, in the sense of the partial order given in [MR20]. Suppose that Ω has exactly n distinct maximal end classes. Then for an arbitrarily large witness $W \subset \Omega$ there exists a n-partite relation Γ_W on $\pi_0(\partial W)$ such that, by extending arcs, $\mathcal{A}(W, \Gamma_W)$ quasi-isometrically embeds in $\mathcal{G}(\Omega)$. Hence choosing an exhaustion of Ω by connected witnesses, we have the following:

[1] Vokes actually requires a natural action of $\mathrm{Mod}(\Sigma)$; however, any subgroup containing $\mathrm{PMod}(\Sigma)$ suffices for the arguments in [Vok22].

Proposition 1.11. *Suppose that Ω has exactly 2 distinct maximal end classes and infinite genus. There exists an asymphoric hierarchically hyperbolic space of arbitrarily large rank (see below) that quasi-isometrically embeds into $\mathcal{G}(\Omega)$.* $\square$

More generally, the proposition also holds whenever Ω has infinitely many non-maximal ends. By [BHS21, Thm. 1.15], it follows that $\operatorname{asdim} \mathcal{G}(\Omega) = \infty$ whenever Ω has exactly two maximal end classes and infinite genus or infinitely many non-maximal ends; we note this fact may be likewise shown by explicitly constructing quasi-flats.

1.1.3. *Hierarchical hyperbolicity.* Behrstock, Hagen, and Sisto introduced hierarchically hyperbolic spaces in [BHS17] to provide a common geometric framework within which to study mapping class groups and cubical groups. In effect, hierarchical hyperbolicity provides an axiomatization and generalization of the geometric structure of the mapping class groups of finite type surfaces elucidated in [MM00]; we direct the reader to [BHS19] for a full enumeration of the axioms, and to [Sis17] for a non-technical discussion of hierarchical hyperbolicity and a survey of results.

We provide a brief description. Let X be a quasi-geodesic space. A hierarchically hyperbolic structure $(X, \mathcal{G})$ on X is comprised of the following:

- an index set $\mathcal{G}$ and a collection of uniformly δ-hyperbolic spaces $\{\mathcal{C}W : W \in \mathcal{G}\}$ with associated projections $\pi_W : X \to 2^{\mathcal{C}W} \setminus \{\varnothing\}$;
- relations $\sqsubseteq$ (*nesting*) and $\perp$ (*orthogonality*) on $\mathcal{G}$;
- if $U \sqsubseteq V$ then a map $\rho^V_U : \mathcal{C}V \to 2^{\mathcal{C}U}$ and if additionally $U \neq V$, a set $\rho^U_V \subset \mathcal{C}V$. If not $U \perp V$ and U, V are not $\sqsubseteq$-comparable, then a set $\rho^U_V \subset \mathcal{C}V$ and *vice versa*.

In addition, the above must satisfy nine axioms for $(X, \mathcal{G})$ to be hierarchically hyperbolic. We consider the model example: the marking complex $\mathcal{MC}(\Sigma)$ defined in [MM00], on which $\operatorname{Mod}(\Sigma)$ acts geometrically, is hierarchically hyperbolic with respect to the index set $\mathcal{G} = \{\text{essential subsurfaces}\}$ and subsurface projections π_W, with nesting and orthogonality determined by inclusion and disjointness, respectively; ρ^V_U is determined by subsurface projection, or the projection of boundary components, as appropriate [BHS19, Thm. 11.1].

By defining a hierarchically hyperbolic structure analogously for partial marking graphs, many of the axioms follow from the fact that they hold for $\mathcal{MC}(\Sigma)$. We omit these here and enumerate two that we will check explicitly:

Ax. 1: *(Projections.)* For $W \in \mathcal{G}$, π_W must be uniformly coarsely Lipschitz and have uniformly quasi-convex image.

Ax. 9: *(Uniqueness.)* For every $K \geq 0$, there exists $K' \geq 0$ such that if $u, v \in X$ such that $d_W(u, v) \leq K$ for all $W \in \mathcal{G}$, then $d(u, v) \leq K'$.

The *rank* ν of a hierarchically hyperbolic space $(X, \mathcal{G})$ is the largest cardinality of a collection of pairwise orthogonal elements $\{U_j\}$ in $\mathcal{G}$ for which $\pi_{U_j}(X)$ is unbounded; if $(X, \mathcal{G})$ is *asymphoric*, then there exists $C \geq 0$ such that ν is likewise the maximum cardinality of a pairwise orthogonal set for which $\operatorname{diam} \mathcal{C}U_j > C$. δ-hyperbolic spaces are hierarchically hyperbolic with respect to a trivial (hence rank at most 1) hierarchical structure; conversely, any rank 1 hierarchically hyperbolic space is δ-hyperbolic if it is also asymphoric [BHS21, Cor. 2.15].

For $\mathcal{A}$ an admissible multiarc and curve graph and a witness W, the action of $\mathrm{PMod}(\Sigma)$ implies $\pi_W(\mathcal{A})$ is either unbounded or $\mathcal{C}W$ is diameter at most 2, if W is disconnected. Hence Theorem 1.3 implies that admissible multiarc and curve graphs are asymphoric hyperbolically hierarchical spaces, and the above and that orthogonality is equivalent to disjointness implies that Conjecture 1.6 holds.

Let $d_W(a,b) := \mathrm{diam}_{\mathcal{C}W}(\pi_W(a) \cup \pi_W(b))$. As with $\mathcal{MC}(\Sigma)$, hierarchically hyperbolic spaces have a distance formula [BHS19, Thm. 4.5]:

Theorem 1.12 (Behrstock-Hagen-Sisto)**.** *Let $(X, \mathcal{G})$ be hierarchically hyperbolic. Then there exists C' such that for any $C > C'$, there exist $K, E \geq 0$ such that*

$$d(a,b) \stackrel{K,E}{=} \sum_{W \in \mathcal{G}} [d_W(a,b)]_C \,.$$

Conjecture 1.7 follows in the admissible case from Theorem 1.12 and Theorem 1.3. //

2. Hierarchical hyperbolicity of partial marking graphs

We begin with a minor generalization of the complete, clean markings defined in [MM00].

Definition 2.1. A *marking* $\mu = \{(a_i, t_i)\}$ on Σ is an essential simple multicurve $\{a_i\}$, denoted $\mathrm{base}\,\mu$, along with a collection of (possibly empty) diameter 1 subsets $t_i \subset \mathcal{C}(a_i)$, denoted $\mathrm{trans}\,\mu$; for $a_i \in \mathrm{base}\,\mu$, let $\mathrm{trans}(\mu, a_i) = t_i$ denote the associated *transversal.*

We say b is a *clean transverse curve* for a curve a if the subsurface F filled by $a \cup b$ has complexity $\xi = 1$ and a, b are adjacent in $\mathcal{C}F$.

Definition 2.2. A marking μ is *locally complete* if whenever $t_i \neq \varnothing$ then the complementary component of $\mathrm{base}\,\mu \setminus \{a_i\}$ containing a_i has complexity $\xi = 1$; if in addition $t_i \neq \varnothing$ implies $t_i = \pi_{a_i} b_i$ for some clean transverse curve b_i intersecting $\mathrm{base}\,\mu$ only in a_i, then μ is *(locally) clean.*

Remark 2.3. A locally clean marking is exactly one whose restriction to the maximal subsurface intersecting only components with non-empty transversals is a complete, clean marking on that subsurface, in the original sense of [MM00]. When we say a marking is *clean*, we will always mean locally clean.

Just as with complete markings, given a locally complete marking μ, a locally clean marking μ' is *compatible* with μ if $\mathrm{base}\,\mu = \mathrm{base}\,\mu'$ and, for all $a \in \mathrm{base}\,\mu$, $\mathrm{trans}(\mu', a) = \varnothing$ if and only if $\mathrm{trans}(\mu, a) = \varnothing$ and $d_a(\mathrm{trans}(\mu, a), \mathrm{trans}(\mu', a))$ is minimal among all possible choices of transversal. We have the following from Remark 2.3 and [MM00, Lem. 2.4]:

Lemma 2.4. *For any locally complete marking μ, there exist at least one and at most 4^b compatible clean markings μ', where b is the number of components in μ with non-empty transversal. Furthermore, for any such μ' and $a \in \mathrm{base}\,\mu$, $d_a(\mathrm{trans}(\mu, a), \mathrm{trans}(\mu', a)) \leq 3$.* □

Definition 2.5. Let $\mu = \{(a_j, t_j)\}, \nu = \{(b_k, s_k)\}$ be two markings on Σ. Then their *geometric intersection number* $i(\mu, \nu)$ is defined as follows:

$$\begin{aligned} i(\mu,\nu) := i(\operatorname{base}\mu, \operatorname{base}\nu) \\ &+ \sum_j \operatorname{diam}_{\mathcal{C}(a_j)}\left[\pi_{a_j}(\operatorname{base}\nu) \cup t_j\right] + \sum_k \operatorname{diam}_{\mathcal{C}(b_k)}\left[\pi_{b_k}(\operatorname{base}\mu) \cup s_k\right] \\ &+ \sum_{a_j = b_k} \operatorname{diam}_{\mathcal{C}(a_j)}(t_j \cup s_k) \end{aligned}$$

Definition 2.6. A *partial marking graph* $\mathcal{M}$ on Σ is a simplicial graph whose vertices are clean markings. If in addition

(i) $\mathcal{M}$ is connected;
(ii) has a natural $\operatorname{PMod}(\Sigma)$ action; and
(iii) there exists $L \geq 0$ such that $(\mu, \nu) \in E(\mathcal{M})$ only if $i(\mu, \nu) \leq L$,

then $\mathcal{M}$ is *admissible*.

We note that admissible multicurve graphs are likewise admissible partial marking graphs; we need only endow each multicurve with empty transversals. The 1-skeleton of the marking complex $\mathcal{MC}(\Sigma)$ in [MM00] is an admissible partial marking graph: its vertex set consists of complete clean markings.

Definition 2.7. Let $\mathcal{M}$ be a partial marking graph on Σ. An essential subsurface $W \subset \Sigma$ is a *witness* for $\mathcal{M}$ if $W \ncong \Sigma_0^3$ and for every marking $\mu \in V(\mathcal{M})$ either (i) W intersects $\operatorname{base}\mu$ or (ii) W has an annular component whose core is a component in $\operatorname{base}\mu$ with non-empty transversal.

Subsurface projection extends to markings, as defined in [MM00]:

Definition 2.8 (Subsurface projection)**.** Let $\mathcal{M}$ be a partial marking graph on Σ, and let $S \subset \Sigma$ be a connected essential subsurface. We define $\pi_S : \mathcal{M} \to 2^{\mathcal{C}S}$ as follows: for $\mu \in V(\mathcal{M})$, if S is an annulus with core $\alpha \in \operatorname{base}\mu$, then $\pi_S(\mu) := \operatorname{trans}(\mu, \alpha)$. Otherwise, $\pi_S(\mu) := \pi_S(\operatorname{base}\mu)$ is the usual multicurve subsurface projection. For S the disjoint union of connected essential subsurfaces S_j, let $\pi_S(\mu) := \bigcup_j \pi_{S_j}(\mu) \subset \mathcal{C}S$.

Remark 2.9. We note that $\pi_W(\mu) \neq \varnothing$ for all $\mu \in V(\mathcal{M})$ if and only if W is a witness for $\mathcal{M}$. $\operatorname{diam}_{\mathcal{C}W}(\pi_W(\mu)) \leq 2$ by *e.g.* [MM00, Lem. 2.3]. For convenience, we shall denote $d_W(\mu, \nu) := \operatorname{diam}_{\mathcal{C}W}(\pi_W(\mu) \cup \pi_W(\nu))$.

Theorem 2.10. *Let $\mathcal{M}$ be an admissible partial marking graph, and let $\mathscr{X}$ denote its collection of witnesses. Then $(\mathcal{M}, \mathscr{X})$ is a hierarchically hyperbolic space with respect to subsurface projection to witness curve graphs $\mathcal{C}W$, $W \in \mathscr{X}$.*

We follow the strategy in [Vok22], extending as necessary to our setting. We first show that the quasi-isometry type of an admissible partial marking graph $\mathcal{M}$ is fully determined by its set of witnesses $\mathscr{X}$; in particular, there exists a canonical "maximal" representative $\mathcal{M} \xrightarrow{\text{q.i.}} \mathcal{L}_{\mathscr{X}}$, which we then conclude to have the desired hierarchically hyperbolic structure.

2.1. **A universal partial marking graph.** We establish a bijection between the coarsely PMod(Σ)-equivariant quasi-isometry types of admissible partial marking graphs on Σ and certain collections of connected essential subsurfaces of Σ.

Definition 2.11. A set $\mathscr{X}$ of essential, non-pants subsurfaces of Σ is an *admissible witness set* for Σ if it is closed under enlargement and the action of PMod(Σ). Let the *connected witness set* $\hat{\mathscr{X}} \subset \mathscr{X}$ denote the connected subsurfaces in $\mathscr{X}$.

The collection of witnesses for an admissible partial marking graph $\mathcal{M}$ is an admissible witness set, denoted $\mathscr{X}^{\mathcal{M}}$. More generally, any collection of essential subsurfaces may be closed to a admissible witness set.

Theorem 2.12. *The map* $\mathcal{M} \mapsto \mathscr{X}^{\mathcal{M}}$ *induces a bijection* $\Psi : [\mathcal{M}] \mapsto \hat{\mathscr{X}}^{\mathcal{M}}$ *between coarsely* PMod(Σ)*-equivariant quasi-isometry types of admissible partial marking graphs and admissible connected witness sets.*

That the map Ψ is well defined will follow from the fact that witness subsurface projections of admissible marking graphs are Lipschitz, as we will see in Section 2.1.2. To show that Ψ is bijective, we construct an admissible partial marking graph $\mathcal{L}_{\hat{\mathscr{X}}}$ for any admissible witness set $\mathscr{X}$ such that (i) $\hat{\mathscr{X}}^{\mathcal{L}_{\hat{\mathscr{X}}}} = \hat{\mathscr{X}}$, and (ii) for any admissible marking graph $\mathcal{M}$ satisfying $\hat{\mathscr{X}}^{\mathcal{M}} = \hat{\mathscr{X}}$, there is a coarsely PMod(Σ)-equivariant quasi-isometry $\mathcal{M} \to \mathcal{L}_{\hat{\mathscr{X}}^{\mathcal{M}}}$.

We extend the notion of an *elementary move* on a clean marking, as introduced in [MM00]:

Definition 2.13. Let μ be a clean marking on Σ, and suppose a clean marking μ' is obtained from μ by either

(i) *a twist:* replace some component $(a, \pi_a b) \in \mu$ with $(a, \pi_a \tau_a b)$, for τ_a a Dehn twist or a half Dehn twist about a.
(ii) *a flip:* replace some component $(a, \pi_a b) \in \mu$ with $(b, \pi_b a)$ and choose a clean marking compatible with the result.

Then μ' is obtained from μ by an *elementary move.* (We assume $i(a, b) > 0$.)

Remark. While not canonical in a strict sense, by Lemma 2.4 flip moves are unique up to finitely many choices for fixed a, b, all uniformly close after projection to witnesses.

Definition 2.14. Let $\mathscr{X}$ be an admissible witness set on Σ. Define $\mathcal{L}_{\mathscr{X}}$ to be the simplicial graph whose vertices are all clean markings that meet every surface in $\mathscr{X}$, in the sense of Definition 2.7, and for which $(\mu, \nu) \in E(\mathcal{L}_{\mathscr{X}})$ if and only if μ is obtained from ν by either

(i) adding or removing a component (a, t), or
(ii) adding or removing a transversal, or
(iii) an elementary move.

Let $\mathcal{L}_{\hat{\mathscr{X}}}$ be defined analogously, replacing $\mathscr{X}$ with $\hat{\mathscr{X}}$.

Lemma 2.15. *Let* $\mathscr{X}$ *be an admissible witness set.* $\mathscr{X}^{\mathcal{L}_{\mathscr{X}}} = \mathscr{X}$ *and* $\hat{\mathscr{X}}^{\mathcal{L}_{\hat{\mathscr{X}}}} = \hat{\mathscr{X}}$.

Proof. $\mathscr{X} \subset \mathscr{X}^{\mathcal{L}_{\mathscr{X}}}$ since by definition the vertices of $\mathcal{L}_{\mathscr{X}}$ meet every subsurface in $\mathscr{X}$, and likewise $\hat{\mathscr{X}} \subset \mathscr{X}^{\mathcal{L}_{\hat{\mathscr{X}}}}$ hence $\hat{\mathscr{X}} \subset \hat{\mathscr{X}}^{\mathcal{L}_{\hat{\mathscr{X}}}}$. Conversely, suppose $W \notin \mathscr{X}$ is an essential, non-pants subsurface of Σ. $\mathscr{X}$ is closed under enlargement, hence W contains no subsurfaces in $\mathscr{X}$. Let μ be the clean marking on Σ obtained from any complete, clean marking on $\Sigma \backslash W$ by adding distinct components in ∂W with empty transversals. Then $\mu \in V(\mathcal{L}_{\mathscr{X}})$ and does not meet W. $W \notin \mathscr{X}^{\mathcal{L}_{\mathscr{X}}}$. Likewise, if $W \notin \hat{\mathscr{X}}$ is an essential connected non-pants subsurface, then $W \notin \hat{\mathscr{X}}^{\mathcal{L}_{\hat{\mathscr{X}}}}$. $\square$

It is clear that $\mathrm{PMod}(\Sigma)$ acts naturally on $\mathcal{L}_{\mathscr{X}}$ and $\mathcal{L}_{\hat{\mathscr{X}}}$ and that, in either graph, two markings are adjacent only if their geometric intersection is less than some uniform constant L. We prove below that $\mathcal{L}_{\mathscr{X}}$ and $\mathcal{L}_{\hat{\mathscr{X}}}$ are connected, hence admissible. In addition, $\mathcal{L}_{\hat{\mathscr{X}}}$ is universal in the following sense: suppose that $\mathcal{M}$ is an admissible marking graph such that $\mathscr{X}^{\mathcal{M}} \supset \hat{\mathscr{X}}$. Then every marking in $\mathcal{M}$ meets every witness in $\hat{\mathscr{X}}$, hence $V(\mathcal{M}) \subset V(\mathcal{L}_{\hat{\mathscr{X}}})$ and is preserved by the action $\mathrm{PMod}(\Sigma)$; we show below that this inclusion is Lipschitz, hence induces a $\mathrm{PMod}(\Sigma)$-equivariant coarse Lipschitz map $\hat{\iota} : \mathcal{M} \to \mathcal{L}_{\hat{\mathscr{X}}}$. An identical argument gives a $\mathrm{PMod}(\Sigma)$-equivariant coarse Lipschitz map $\iota : \mathcal{M} \to \mathcal{L}_{\mathscr{X}}$ whenever $\mathscr{X}^{\mathcal{M}} \supset \mathscr{X}$, in which case $\hat{\iota}$ factors coarsely (exactly on vertices) as the composition of the universal maps $\mathcal{M} \to \mathcal{L}_{\mathscr{X}} \to \mathcal{L}_{\hat{\mathscr{X}}}$ induced by the inclusions $\mathscr{X}^{\mathcal{M}} \supset \mathscr{X} \supset \hat{\mathscr{X}}$.

We note that $\mathcal{MC}(\Sigma)^{(1)}$ admits every essential, non-pants subsurface as a witness, hence $\mathscr{X}(\mathcal{MC}(\Sigma)^{(1)}) \supset \mathscr{X} \supset \hat{\mathscr{X}}$. Since the edges of $\mathcal{MC}(\Sigma)^{(1)}$ correspond to elementary moves, ι may be taken to be a (simplicial) embedding and $\mathcal{MC}(\Sigma)^{(1)}$ as a connected $\mathrm{PMod}(\Sigma)$-invariant full subgraph of $\mathcal{L}_{\mathscr{X}}$, which is in turn a $\mathrm{PMod}(\Sigma)$-invariant full subgraph of $\mathcal{L}_{\hat{\mathscr{X}}}$. Hence to show that $\mathcal{L}_{\mathscr{X}}$ and $\mathcal{L}_{\hat{\mathscr{X}}}$ are connected, it suffices to prove that every marking lies in the same component as $\mathcal{MC}(\Sigma)^{(1)}$, as the following lemma shows:

Lemma 2.16. *Any marking μ in $V(\mathcal{L}_{\mathscr{X}})$ or $V(\mathcal{L}_{\hat{\mathscr{X}}})$ may be completed to a complete, clean marking μ' through a sequence of length at most $2\xi(\Sigma) - |\mu|$ of adding components and transversals.*

Proof. We note that we may always add components with empty transversal: since μ is already locally complete, no new base curve intersects existing clean transverse curves. Likewise, we may always add (clean) transversals to complete markings. Add $\xi(\Sigma) - |\mu|$ components with empty transversal to obtain the complete marking μ''. For at most $\xi(\Sigma)$ components with empty transversal in μ'', add transversals to obtain a complete, clean marking $\mu' \in \mathcal{MC}(\Sigma)^{(1)}$. $\square$

2.1.1. *The quasi-isometry.* We restate the arguments in [Vok22], with a substantial adapatation of the proof of the existence of a quasi-retraction for our setting. Let $\mathcal{M}$ be an admissible partial marking graph on Σ with witness set $\mathscr{X}$, and let $\hat{\iota} : \mathcal{M} \to \mathcal{L}_{\hat{\mathscr{X}}}$ be the universal coarse map from above. We show that $\hat{\iota}$ is a quasi-isometry.

We will use a key feature of admissibility, namely the existence of an upper bound on distance in terms of geometric intesection number:

Lemma 2.17. *Any admissible partial marking graph $\mathcal{M}$ admits a monotonic function $f_{\mathcal{M}} : \mathbb{N} \to \mathbb{N}$ such that for all $\mu, \nu \in V(\mathcal{M})$, $d(\mu, \nu) \leq f_{\mathcal{M}}(i(\mu, \nu))$.*

In fact, Lemma 2.17 follows only from connectivity and the natural action of a finite index subgroup $H \leq \mathrm{Mod}(\Sigma)$: for any $M \geq 0$, $\mathrm{Mod}(\Sigma)$ acts cofinitely on the set of pairs of clean markings with geometric intesection less than M. However, given that $(\mu, \nu) \in E(\mathcal{M})$ only if $i(\mu, \nu) \leq L$, it is immediate that $\hat{\iota}$ is $f_{\mathcal{L}(\hat{\mathscr{X}})}(L)$-Lipschitz; we remark that this argument implies that the universal maps $\hat{\iota} : \mathcal{M} \to \mathcal{L}_{\hat{\mathscr{X}}}$ and $\iota : \mathcal{M} \to \mathcal{L}_{\mathscr{X}}$ are always Lipshitz, even if $\mathscr{X}^{\mathcal{M}} \supsetneq \mathscr{X}$.

Coarse surjectivity likewise results from Lemma 2.17 and the following:

Lemma 2.18. *There exists $M \geq 0$ such that for any $\mu \in V(\mathcal{L}_{\hat{\mathscr{X}}})$, there exists $\nu \in V(\mathcal{M})$ such that $i(\mu, \nu) \leq M$.*

Remark 2.19. Lemma 2.18 is a special case of following general fact: for any G-invariant function $s : A \times B \to \mathbb{N}$ on cofinite G-sets A, B, $\min_{b \in B} s(a, b)$ is uniformly bounded for $a \in A$. In fact, it follows that for any cofinite G-set A, there exists $C \geq 0$ such that for any $a, a' \in A$, there exists $g \in G$ such that $s(a, ga') \leq C$: let C be the maximum of the bounds obtained for $B \in A/G$. We will use this fact below.

Finally, we construct a coarse Lipschitz retraction $\rho : \mu \mapsto E_\mu$, where $E_\mu = \{\nu \in V(\mathcal{M}) : i(\mu, \nu) \leq M\}$ and M is the constant in Lemma 2.18. By Lemma 2.17, it is clear that $\hat{\iota}\rho$ is coarsely identity. Moreover, we note that $\mathrm{PMod}(\Sigma)$ acts cofinitely on pairs of adjacent vertices $(\mu, \mu') \in E(\mathcal{L}_{\hat{\mathscr{X}}})$ since $i(\mu, \mu') < L$, and that ρ is $\mathrm{PMod}(\Sigma)$-equivariant, hence $d_{\mathcal{M}}(E_\mu, E_{\mu'})$ is uniformly bounded. It remains only to check the following:

Lemma 2.20. $\mathrm{diam}_{\mathcal{M}}(E_\mu)$ *is uniformly bounded.*

We must first build some machinery. Let $K \subset \Sigma$ be an essential, non-peripheral, and possibly disconnected subsurface. Given ω a clean marking on Σ, fix a representative in minimal position with ∂K and let $\omega|_K := (\sigma, \alpha)$, where $\sigma \subset \omega$ is the submarking comprised of components whose representative curve is fully contained in K, and α is an arc system obtained from the remaining components by taking the intersection of their representative curves with K. We regard $\omega|_K$ up to isotopy, rel ∂K. Let $i_K(\omega|_K, \omega'|_K) := i(\sigma, \sigma') + i(\alpha, \sigma') + i(\sigma, \alpha') + i(\alpha, \alpha')$, where $\omega|_K = (\sigma, \alpha)$ and $\omega'|_K = (\sigma', \alpha')$ and the geometric intersection number between arc systems and markings is defined as follows:

$$i(\nu, \alpha) := i(\mathrm{base}\,\nu, \alpha) + \sum_{(a,t) \in \nu} \mathrm{diam}_{\mathcal{C}(a)}(t \cup \pi_a \alpha)$$

We emphasize that $\omega|_K$ is well defined only up to choice of representative isotopic rel ∂K, which we assume to be fixed for given ω unless otherwise specified. $\mathrm{Mod}(K, \partial K)$ acts on the set of pairs $\omega|_K$ and preserves i_K.

Remark. Here our notation differs with that of [MM00], where $\omega|_K$ instead denotes the *restriction* of ω to K.

Claim 2.21. *There exists $D \geq 0$ such that for any $\omega|_K, \omega'|_K$ with $i(\omega, \partial K), i(\omega', \partial K)$ at most M, there exists $\phi \in \mathrm{Mod}(K, \partial K)$ for which $i_K(\phi\omega|_K, \omega'|_K) \leq D$.*

Proof. Let $\mathscr{S}$ be the set of equivalence classes of pairs $\eta|_K$ for some choice of representative of $\eta \in V(\mathcal{M})$ with $i(\eta, \partial K) \leq M$, up to (endpoint free) isotopy and ignoring transversals on components peripheral in K. $\mathrm{PMod}(K)$ acts cofinitely on $\mathscr{S}$ and preserves

$$i_K([\eta|_K],[\eta'|_K]) := \min_{\rho|_K\in[\eta|_K],\rho'|_K\in[\eta'|_K]} i_K(\rho|_K,\rho'|_K)$$

hence likewise does $\mathrm{Mod}(K,\partial K)\twoheadrightarrow \mathrm{PMod}(K)$. By Remark 2.19, there exists D' independent of $\omega|_K,\omega'|_K$ and $\psi\in\mathrm{Mod}(K,\partial K)$ such that $i_K([\psi\omega|_K],[\omega'|_K])\le D'$. However, we observe that if $[\eta|_K]=[\eta'|_K]$, then there exists D'' depending only on M and some boundary multitwist $\tau\in\mathrm{Mod}(K,\partial K)$ such that $i_K(\tau\eta|_K,\gamma|_K)\le i_K(\eta'|_K,\gamma|_K)+D''$ for all $\gamma|_K$, whence the claim follows with $D=D'+2D''$. □

Claim 2.22. *Let η be a multicurve on Σ and $K_l\subset\Sigma$ pairwise disjoint subsurfaces partitioning Σ whose boundary curves lie in η. There exists $C\ge 0$ such that, for any markings ω,ω' with $i(\omega,\eta),i(\omega',\eta)\le M$, $i(\omega,\omega')\le\sum_l i_{K_l}(\omega|_{K_l},\omega'|_{K_l})+C$.*

Proof. Let $\gamma=\{(a_j,t_j)\},\gamma'=\{(a'_k,t'_k)\}$ be the maximal submarkings of ω,ω' respectively whose base curves all intersect η. Then by our definition of i_{K_l}, we have that

$$\begin{aligned} i(\omega,\omega') \le \sum_l i_{K_l}(\omega|_{K_l},\omega'|_{K_l}) &+ \sum_{(a_j,t_j)\in\gamma} \mathrm{diam}_{\mathcal{C}(a_j)}(t_j\cup\pi_{a_j}(\mathrm{base}\,\omega')) \\ &+ \sum_{(a'_k,t'_k)\in\gamma'} \mathrm{diam}_{\mathcal{C}(a'_k)}(t'_k\cup\pi_{a'_k}(\mathrm{base}\,\omega)) \\ &+ \sum_{a_j=a'_k} \mathrm{diam}_{\mathcal{C}(a_j)}(t_j\cup t'_k)\,. \end{aligned}$$

It suffices to show that the (at most $3\xi(\Sigma)$) transversal terms are uniformly bounded. Suppose $(a_j,t_j)\in\gamma$. a_j intersects η, hence $\pi_{a_j}\eta\ne\varnothing$. Since $i(\omega,\eta),i(\omega',\eta)\le M$, $\mathrm{diam}_{\mathcal{C}(a_j)}(t_j\cup\pi_{a_j}\eta)\le M$ and $\mathrm{diam}_{\mathcal{C}(a_j)}(\pi_{a_j}\eta\cup\pi_{a_j}(\mathrm{base}\,\omega'))$ is uniformly bounded, hence $\mathrm{diam}_{\mathcal{C}(a_j)}(t_j\cup\pi_{a_j}(\mathrm{base}\,\omega'))$ is uniformly bounded. Analogous arguments apply for $(a'_k,t'_k)\in\gamma'$ and when $a_j=a'_k$. □

Proof of Lemma 2.20. Fix a representative of μ and let N be an open regular neighborhood, and let K_j denote the complementary components of N along with the components of $\overline{N}$. Let $K^j:=\Sigma\setminus\mathring{K}_j$. Fix representatives of ν,ν' in minimal position with ∂K_j for all j; since $\nu,\nu'\in E_\mu$, the number of arc components in $\nu\cap K_j$ and $\nu'\cap K_j$ is at most M, hence whenever $K_j\cong\Sigma_0^3$ we may assume that $i_{K_j}(\nu|_{K_j},\nu'_{K_j})$ is uniformly bounded up to isotoping intersections into surrounding annuli. Let $\nu_0=\nu$. For each K_j, we will construct $\nu_j\in V(\mathcal{M})$ such that ν_j is identical to ν_{j-1} except in $\mathring{K}_j$ (hence $i(\nu_j,\mathrm{base}\,\mu)\le M$) and $d_{\mathcal{M}}(\nu_j,\nu_{j-1})$ and $i_{K_j}(\nu_j|_{K_j},\nu'|_{K_j})$ are uniformly bounded. The lemma then follows from Claim 2.22.

We construct ν_j inductively: assume ν_{j-1} exists and suppose that $K_j\notin\hat{\mathscr{X}}$, hence $K_j\notin\mathscr{X}$ since K_j is connected. If $K_j\cong\Sigma_0^3$ then it suffices to let $\nu_j=\nu_{j-1}$. Else, there exists $\omega\in V(\mathcal{M})$ disjoint from K_j. Applying Claim 2.21, up to translation by $\mathrm{Mod}(K^j,\partial K^j)$ we may assume that $i_{K^j}(\omega|_{K^j},\nu_{j-1}|_{K^j})\le D$; likewise, choose $\phi\in\mathrm{Mod}(K_j,\partial K_j)$ such that $i_{K_j}(\phi\nu_{j-1}|_{K_j},\nu'|_{K_j})\le D$ and, extending ϕ by identity, let $\nu_j=\phi\nu_{j-1}$. Then $i_{K_j}(\nu_j|_{K_j},\nu'|_{K_j})\le D$. ω is disjoint from K_j, hence $\omega|_{K_j}=\varnothing$ and $i_{K_j}(\nu_{j-1}|_{K_j},\omega|_{K_j})=0$ and by Claim 2.22, $i(\nu_{j-1},\omega)\le$

$D + C$. ν_j, ν_{j-1} are identical outside of K_j, thus likewise $i(\nu_j, \omega) \leq D + C$. Hence $d_{\mathcal{M}}(\nu_j, \nu_{j-1}) \leq d_{\mathcal{M}}(\nu_j, \omega) + d_{\mathcal{M}}(\omega, \nu_{j-1}) \leq 2 f_{\mathcal{M}}(D + C)$.

If $K_j \in \hat{\mathscr{X}}$, then K_j is an annulus; let $a_j \in \operatorname{base} \mu$ be its core curve. Since K_j is a witness, $\operatorname{trans}(\mu, a_k) \neq \varnothing$. $i(\mu, \nu'), i(\mu, \nu_{j-1}) \leq M$ then implies that $i_{K_j}(\nu_{j-1}|_{K_j}, \nu'|_{K_j})$ is uniformly bounded. Let $\nu_j = \nu_{j-1}$. □

Remark. The universal map $\iota : \mathcal{M} \to \mathcal{L}_{\mathscr{X}}$ is also a PMod(Σ)-equivariant quasi-isometry. In particular, the universal map $\iota' : \mathcal{L}_{\mathscr{X}} \to \mathcal{L}_{\hat{\mathscr{X}}}$ is a PMod(Σ)-equivariant quasi-isometry by the above and $\hat{\iota} : \mathcal{M} \to \mathcal{L}_{\hat{\mathscr{X}}}$ factors through ι and ι'. //

2.1.2. *Witness subsurface projections are Lipschitz.* We consider $\pi_W : \mathcal{L}_{\hat{\mathscr{X}}} \to 2^{\mathcal{C}W}$ for $W \in \hat{\mathscr{X}}$; the arguments for $\mathcal{L}_{\mathscr{X}}$ are identical. It suffices that for $(\mu, \nu) \in E(\mathcal{L}_{\hat{\mathscr{X}}})$ and $W \in \hat{\mathscr{X}}$, $d_W(\mu, \nu) = \operatorname{diam}_{\mathcal{C}W}(\pi_W(\mu) \cup \pi_W(\nu))$ is uniformly bounded. If μ, ν differ by the addition of a component or a transversal, then without loss of generality $\pi_W(\mu) \subset \pi_W(\nu)$ and $d_W(\mu, \nu) = \operatorname{diam}_{\mathcal{C}W}(\pi_W(\nu)) \leq 2$. Otherwise, μ, ν differ by an elementary move. Since μ, ν are locally complete and clean, we may apply the proof of Lemma 2.5 in [MM00]:

Proposition 2.23 (Masur-Minsky)**.** *Suppose $\mu, \nu \in V(\mathcal{L}_{\hat{\mathscr{X}}})$ and ν is obtained from μ by an elementary move. Then for any $W \in \hat{\mathscr{X}}$, $d_W(\mu, \nu) \leq 4$.*

We may now prove that the map $[\mathcal{M}] \mapsto \hat{\mathscr{X}}^{\mathcal{M}}$ from Theorem 2.12 is well defined. Let $\hat{\mathscr{X}} = \hat{\mathscr{X}}^{\mathcal{M}}$. Since $V(\mathcal{M}) \subset V(\mathcal{L}_{\hat{\mathscr{X}}})$, the extension $\hat{\iota} : \mathcal{M} \to \mathcal{L}_{\hat{\mathscr{X}}}$ preserves subsurface projection; since $\hat{\iota}$ is a quasi-isometry, witness subsurface projections $\pi_W : \mathcal{M} \to 2^{\mathcal{C}W}$, $W \in \hat{\mathscr{X}}$ are likewise Lipschitz. Suppose that $\mathcal{M}, \mathcal{M}'$ have distinct connected witness sets $\hat{\mathscr{X}}, \hat{\mathscr{X}}'$ respectively, and let $W \in \hat{\mathscr{X}}' \setminus \hat{\mathscr{X}}$. Then there exists $\omega \in V(\mathcal{M})$ disjoint from W. Since W is connected, we may choose a loxodromic element $\varphi \in \operatorname{Mod}(W, \partial W)$ for $\mathcal{C}W$, any extension of which acts on $\mathcal{M}'$ with non-zero translation length since $\pi_W : \mathcal{M}' \to 2^{\mathcal{C}W}$ is Lipschitz. Let $\tilde{\varphi} \in \operatorname{PMod}(\Sigma)$ be an extension by identity. Then $\tilde{\varphi}$ fixes $\omega \in \mathcal{M}$, hence $\mathcal{M}, \mathcal{M}'$ are not PMod(Σ)-equivariantly quasi-isometric. //

Remark. The above implies that $\mathcal{MC}(\Sigma)^{(1)}$, while a $2\xi(\Sigma)$-coarsely dense full subcomplex of $\mathcal{L}_{\mathscr{X}}$ by Lemma 2.16, is in general significantly distorted.

2.2. **Hierarchical structure of $(\mathcal{L}_{\mathscr{X}}, \mathscr{X})$.** We endow $\mathcal{L}_{\mathscr{X}}$ with the usual hierarchical structure via witness subsurface projections $\{\pi_W : \mathcal{L}_{\mathscr{X}} \to 2^{\mathcal{C}W}\}_{W \in \mathscr{X}}$ and the following relations on $\mathscr{X}$:

- $U \sqsubseteq V$ if and only if $U \subset V$ up to isotopy; and
- $U \perp V$ if and only if U, V are disjoint up to isotopy.

Let $U \pitchfork V$ if neither of the above hold. For $W \in \mathscr{X}$, π_W is uniformly quasi-convex: any curve (potentially adding transversal) may be completed to a marking in $V(\mathcal{L}_{\mathscr{X}})$, thus π_W is surjective if W is not an annulus; if W is an annulus with core curve a, then any orbit of a Dehn twist about a projects to a 1-coarsely dense subset of $\mathcal{C}W$. Hence π_W satisfies Axiom 1 by the above, Remark 2.9, and Proposition 2.23.

If $U \sqsubseteq V$, then let $\rho^V_U = \pi_U : \mathcal{C}V \to 2^{\mathcal{C}U}$, and if in addition $U \neq V$, let $\rho^U_V = \partial_V U$, the non-peripheral boundary of U in V. If $U \pitchfork V$, let $\rho^V_U = \pi_U(\partial V)$.

Axioms 2 and 3 follow immediately from the definition of witness subsurfaces[2] and our choice of relations and associated projections, and Axioms 4 through 8 follow from the fact that the same hold for $\mathcal{MC}(\Sigma)^{(1)}$, which lies as a $2\xi(\Sigma)$-coarsely dense subcomplex of $\mathcal{L}_{\mathscr{X}}$, and that the π_W are uniformly Lipschitz. We verify Axiom 9 below.

Definition 2.24. Given an admissible witness set $\mathscr{X}$, let the *twist-free witness set* $\tilde{\mathscr{X}} \subset \mathscr{X}$ denote the admissible witness set with all annular witnesses removed.

Definition 2.25. If $\mathscr{X}$ is an admissible witness set without annuli, let $\mathcal{K}_{\mathscr{X}}$ denote the full subgraph of $\mathcal{L}_{\mathscr{X}}$ spanned by markings with empty transversals, with additional edges (μ, ν) corresponding to *twist-free flip moves*: ν is obtained from μ by replacing a component $(a, \varnothing) \mapsto (b, \varnothing)$ such that, if F is the subsurface filled by a, b, then F is connected, $\partial F \subset \operatorname{base}\mu \cup \partial\Sigma$, and $d_{\mathcal{C}F}(a, b) = 1$.

Lemma 2.26. *For any $K \geq 0$, there exists $K' \geq 0$ such that for any $\mu, \nu \in \mathcal{L}_{\mathscr{X}}$, if $d_W(\mu, \nu) \leq K$ for all $W \in \mathscr{X}$, then $d_{\mathcal{L}_{\mathscr{X}}}(\mu, \nu) \leq K'$.*

Proof. We regard $\mathcal{K}_{\tilde{\mathscr{X}}}$ as a (twist-free) multicurve graph. For any multicurve $m \in V(\mathcal{K}_{\tilde{\mathscr{X}}})$, there exists a clean marking $\mu \in V(\mathcal{L}_{\mathscr{X}})$ such that $\operatorname{base}\mu = m$. In particular every non-annular witness in $\mathscr{X}$ intersects m, and if A is some annular witness disjoint from m with core curve a, then $a \in m$ and any complementary component adjacent to a is a pair of pants. Hence we may choose a clean transverse curve b for a disjoint from $m \setminus \{a\}$. Let μ consist of components $(a, \pi_a b)$ for $a \in m$ if a is parallel to an annular witness, and $(a, \varnothing)$ otherwise.

By [Vok22, Prop. 3.6], there exists a path of multicurves $\tilde{P} \subset \mathcal{K}_{\tilde{\mathscr{X}}}$ between $\operatorname{base}\mu, \operatorname{base}\nu$ of length ℓ at most $K'' = K''(K, \mathscr{X})$. Let $\tilde{P} = (m_j)$ with $m_1 = \operatorname{base}\mu$ and $m_\ell = \operatorname{base}\nu$. Let $P_0 = (\omega_j) \in V(\mathcal{L}_{\mathscr{X}})$ be a sequence of clean markings such that $\omega_1 = \mu, \omega_\ell = \nu$, and $\operatorname{base}\omega_j = m_j$, where $\omega_{j \neq 1,\ell}$ is chosen as described above. If $d_{\mathcal{L}_{\mathscr{X}}}(\omega_j, \omega_{j+1})$ is uniformly bounded indepedent of the path $\tilde{P}$, then we conclude by the triangle inequality. In fact, it suffices to control projections to annular witnesses:

Claim 2.27. *Let $\omega, \omega' \in V(\mathcal{L}_{\mathscr{X}})$ such that $(\operatorname{base}\omega, \operatorname{base}\omega') \in E(\mathcal{K}_{\tilde{\mathscr{X}}})$, and let $D \geq 0$ such that $d_c(\omega, \omega') \leq D$ for any $c \in \operatorname{base}\omega \cup \operatorname{base}\omega'$ parallel to an annular witness. Then $d_{\mathcal{L}_{\mathscr{X}}}(\omega, \omega') \leq (D+3)(\xi(\Sigma)+1)$.*

Proof of claim. Without loss of generality assume $|\omega| \leq |\omega'|$, hence $\operatorname{base}\omega'$ is obtained from $\operatorname{base}\omega$ by either adding a component b or a twist-free flip $a \mapsto b$. Let ω'' be obtained from ω by adding $(b, \operatorname{trans}(\omega', b))$ or replacing $(a, \operatorname{trans}(\omega, a))$ with $(b, \pi_b a)$ and choosing a compatible clean marking, as appropriate. In the first case, $d_{\mathcal{L}_{\mathscr{X}}}(\omega, \omega'') = 1$, and if a is not parallel to an annular witness, then in the second case $d_{\mathcal{L}_{\mathscr{X}}}(\omega, \omega'') \leq 3$: remove and replace the transversal for a with $\pi_a b$, then flip. Otherwise, a is a witness curve and at most $d_a(\omega, \omega') \leq D$ twist moves along a suffice to replace $\operatorname{trans}(\omega, a)$ with $\pi_a b$, hence $d_{\mathcal{L}_{\mathscr{X}}}(\omega, \omega'') \leq D + 1$.

[2]We remark that Axiom 3 may require disconnected witnesses. Thus $(\mathcal{L}_{\hat{\mathscr{X}}}, \hat{\mathscr{X}})$ may not be hierarchically hyperbolic, despite that the quasi-isometry $\hat{\imath} : \mathcal{L}_{\mathscr{X}} \to \mathcal{L}_{\hat{\mathscr{X}}}$ preserves projections over $\hat{\mathscr{X}}$.

We show that $d_{\mathcal{L}_{\mathscr{X}}}(\omega'', \omega') \leq (D+3)|\omega''|$ and conclude. $\operatorname{base}\omega'' = \operatorname{base}\omega'$, hence to obtain ω' we need only modify transversals in ω'' to agree with those in ω'. Let $c \in \operatorname{base}\omega''$. As above, at most 2 moves suffice if c is not a witness curve and $d_c(\omega'', \omega')$ otherwise; by Lemma 2.4, $d_c(\omega'', \omega') \leq d_c(\omega, \omega') + 3 \leq D + 3$. □

Let $P = (\omega_j)$ be a sequence of markings in $\mathcal{L}_{\mathscr{X}}$ of length $\ell \leq K''$ such that $\omega_1 = \mu$, $\omega_\ell = \nu$, $(\operatorname{base}\omega_j)$ is a path in $\mathcal{K}_{\tilde{\mathscr{X}}}$, and P is minimal in the following sense: P has the fewest number of pairs (j, a) for which $1 \leq j < \ell$, $a \in \operatorname{base}\omega_j \cup \operatorname{base}\omega_{j+1}$ is parallel to an annular witness, and $d_a(\omega_j, \omega_{j+1}) > K + c_0K'' + 2$, where c_0 is a universal constant defined below. P exists by the existence of P_0. We show that if P has such a pair (j, a) then there exists a more minimal sequence P', and conclude by contradiction and the above. Fix a in such a pair. Let $m_j = \operatorname{base}\omega_j$. If $a \notin m_j \cup m_{j+1}$, then $d_a(\omega_j, \omega_{j+1}) \leq 4$ (see *e.g.* the proof of [MM00, Lem. 2.5]). For each $j < \ell$ for which $a \in m_j \cup m_{j+1}$, choose $\sigma_j \in \mathbb{Z}$ such that $d_a(\omega_j, \tau_a^{\sigma_j}\omega_{j+1}) \leq 1$, where τ_a is a Dehn twist about a; let $J < \ell$ be the last such index. Set $\sigma_j = 0$ otherwise. $\mathcal{C}(a)$ is $\mathbb{Z}$-equivariantly $(1,1)$-quasi-isometric to $\mathbb{Z}$ and $d_a(\mu, \nu) = d_a(\omega_1, \omega_\ell) \leq K$, hence there exists a universal constant $c_0 > 4$ such that

$$\left|\sum_{j=1}^{J} \sigma_j\right| \leq K + c_0(\ell - 1) + 1 \leq K + c_0K'' \,. \tag{1}$$

Let $\rho_j = \sum_{k=1}^{j-1} \sigma_k$ and let $P' = (\omega'_j)$, where $\omega'_j = \tau_a^{\rho_j}\omega_j$ for $j \leq J$, and $\omega'_j = \omega_j$ otherwise; let $m'_j = \operatorname{base}\omega'_j$. We first verify that (m'_j) is a path in $\mathcal{K}_{\tilde{\mathscr{X}}}$. Since $\mathcal{K}_{\tilde{\mathscr{X}}}$ is preserved by τ_a, for $j < J$ it suffices that $(m_j, \tau_a^{\sigma_j} m_{j+1})$ is an edge in $\mathcal{K}_{\tilde{\mathscr{X}}}$. For $\sigma_j = 0$, this is immediate. Else $a \in m_j \cup m_{j+1}$. If $a \in m_{j+1}$ then $\tau_a^{\sigma_j} m_{j+1} = m_{j+1}$, and otherwise $a \in m_j$ and m_{j+1} is obtained by a twist-free flip $a \mapsto b$: we observe that $\tau_a^{\sigma_j} b$ intersects a minimally in the surface filled by a, b. For $j = J$, it suffices that $(m_J, \tau_a^{-\rho_J} m_{J+1})$ is an edge in $\mathcal{K}_{\tilde{\mathscr{X}}}$: since $a \in m_J \cup m_{J+1}$ by assumption, an identical argument applies.

We show that P' has strictly fewer pairs (j, a') with $a' \in m'_j \cup m'_{j+1}$ a witness curve such that $d_{a'}(\omega'_j, \omega'_{j+1}) > K + c_0K'' + 2$.

Claim 2.28. *Let ω, ω' be clean markings whose respective base multicurves m, m' differ by adding a component or a (twist-free) flip. Let $a \in m \cup m'$ and $a'' \in m$ be distinct. Then $d_{a''}(\omega, \tau_a^q\omega') = d_{a''}(\omega, \omega')$ for $q \in \mathbb{N}$.*

Proof of claim. If $a'' \in m \cap m'$, then $\pi_{a''}(\omega) = \operatorname{trans}(\omega, a'')$ and $\pi_{a''}(\omega') = \operatorname{trans}(\omega', a'')$, projections of clean transverse curves b, c respectively. Since $a \in m \cup m'$, $a \cap a'' = \varnothing$ and at most one of b, c intersects a, hence τ_a lifts to an action on $\mathcal{C}(a'')$ that fixes either $\pi_{a''}(\omega)$ or $\pi_{a''}(\omega')$: we obtain $d_{a''}(\omega, \tau_a^q\omega') = d_{a''}(\tau_a^q\omega, \tau_a^q\omega') = d_{a''}(\omega, \omega')$. If instead $a'' \in m \setminus m'$, then $a \in m'$, since otherwise two distinct curves lie in $m \setminus m'$ and m, m' do not differ by adding a component or a flip. Then $\pi_{a''}(\omega') = \pi_{a''}(m')$ is preserved by τ_a and likewise $d_{a''}(\omega, \tau_a^q\omega') = d_{a''}(\omega, \omega')$. □

Observe that the maps $a' \in m'_j \mapsto \tau_a^{-\rho_j} a'$ and $a' \in m'_{j+1} \mapsto \tau_a^{-\rho_{j+1}} a'$ induce a bijection from $m'_j \cup m'_{j+1}$ to $m_j \cup m_{j+1}$ preserving witness curves. We first assume that $j < J$, and suppose that $a' \in m'_j$ is a witness curve. Let $a'' = \tau_a^{-\rho_j} a' \in m_j$.

As above, by translating by $\tau_a^{-\rho_j}$ we observe that $d_{a'}(\omega'_j, \omega'_{j+1}) = d_{a''}(\omega_j, \tau_a^{\sigma_j}\omega_{j+1})$. If $\sigma_j = 0$ then $d_{a''}(\omega_j, \tau_a^{\sigma_j}\omega_{j+1}) = d_{a''}(\omega_j, \omega_{j+1})$. Suppose $\sigma_j \neq 0$ and thus $a \in m_j \cup m_{j+1}$. For $a \neq a''$, Claim 2.28 implies that likewise $d_{a''}(\omega_j, \tau_a^{\sigma_j}\omega_{j+1}) = d_{a''}(\omega_j, \omega_{j+1})$. An analogous argument applies if $a' \in m'_{j+1}$, replacing a'' with $\tau_a^{-\rho_{j+1}}a' \in m_{j+1}$ and translating by $\tau_a^{-\rho_{j+1}}$. Finally if $a'' = a$ then $a' = a$, thus by our choice of σ_j $d_{a'}(\omega_j, \tau_a^{\sigma_j}\omega_{j+1}) \leq 1 \leq K + c_0K'' + 2$.

Let $j = J$. Then $a \in m_J \cup m_{J+1}$ by our choice of J. We apply the arguments above: if $a' \in m'_J$, then let $a'' = \tau_a^{-\rho_J}a' \in m_J$ and translate by $\tau_a^{-\rho_J}$, else if $a' \in m'_{J+1} = m_{J+1}$, then let $a'' = a'$. Hence if $a'' \neq a$, $d_{a'}(\omega'_J, \omega'_{J+1}) = d_{a''}(\omega_J, \omega_{J+1})$. If $a'' = a$ then $a' = a$; observe that by (1), $|\rho_{J+1}| \leq K + c_0K''$, hence $d_a(\omega_{J+1}, \tau_a^{\rho_{J+1}}\omega_{J+1}) \leq K + c_0K'' + 1$. But $d_a(\omega'_J, \tau_a^{\rho_{J+1}}\omega_{J+1}) = d_a(\omega_J, \tau_a^{\sigma_J}\omega_{J+1}) \leq 1$, hence $d_{a'}(\omega'_J, \omega'_{J+1}) \leq K + c_0K'' + 2$. □

Hence $(\mathcal{L}_{\mathscr{X}}, \mathscr{X})$ is a hierarchically hyperbolic space with respect to witness subsurface projections. However, if $(X, \mathcal{G})$ is a hierarchically hyperbolic space and $\psi : X' \to X$ is a quasi-isometry, then $(X', \mathcal{G})$ is likewise a hierarchically hyperbolic space by precomposing projections with ψ (this is remarked in *e.g.* [BHS17, §1.1.4]). Thus by Section 2.1, Theorem 2.10 follows: for any $\mathcal{M}$ for which $\mathscr{X}^{\mathcal{M}} = \mathscr{X}$, the universal quasi-isometry $\iota : \mathcal{M} \to \mathcal{L}_{\mathscr{X}}$ preserves subsurface projection, hence $(\mathcal{M}, \mathscr{X})$ is a hierarchically hyperbolic space with respect to the projections $\pi_W \circ \iota = \pi_W$, $W \in \mathscr{X}$. //

3. Hierarchical hyperbolicity of multiarc and curve graphs

For an admissible multiarc and curve graph $\mathcal{A}$ on Σ, we construct an admissible partial marking graph $\mathcal{M}_{\mathcal{A}}$ on Σ with an identical witness set and a PMod(Σ)-equivariant coarse quasi-isometry $\zeta : \mathcal{A} \to \mathcal{M}_{\mathcal{A}}$ that coarsely preserves witness subsurface projection. Then by Theorem 2.10, Theorem 1.3 follows immediately.

3.1. **Constructing the associated marking graph.** Given an arc and curve system α, we construct a set of corresponding clean markings μ_α as follows. Let α_j denote the connected components of α, *i.e.* maximal subsystems such that $\alpha_j \cup \partial\Sigma$ has a single connected component that is not a boundary. Then α_j is either: (i) an arc or curve a_j, or (ii) a multiarc $\{a_{j,k}\}_{k=1}^{L_j}$. In the former case, let base $\mu_j = \partial_\Sigma(a)$, the collection of distinct essential, non-peripheral boundary components of a regular neighborhood of $a \cup \partial\Sigma$, and give each component an empty transversal. In the latter, let base $\mu_j = \bigcup_k \partial_\Sigma(\bigcup_{s=1}^k a_{j,s})$ and to each component c add the transversal $\pi_c(\alpha_j)$. Let F_j be the subsurface filled by α_j. Then base $\mu_j \setminus \partial F_j \subset F_j$, $\partial F_j \cap$ base μ_j has empty transversals, and $\mu_j \setminus \partial F_j$ is a complete marking on F_j: $\bigcup_j \mu_j$ is a locally complete marking. Let μ_α be the (finite) collection of all clean markings compatible with $\bigcup_j \mu_j$, for every choice of orderings of the $a_{j,k}$.

We define the vertices of $\mathcal{M}_{\mathcal{A}}$ to be

$$V(\mathcal{M}_{\mathcal{A}}) = \bigcup_{\alpha \in V(\mathcal{A})} \mu_\alpha .$$

Let $(\mu, \nu) \in E(\mathcal{M}_{\mathcal{A}})$ if and only if $\mu \in \mu_\alpha$ and $\nu \in \mu_\beta$ for $(\alpha, \beta) \in E(\mathcal{A})$. Finally, define $\zeta : V(\mathcal{A}) \to V(\mathcal{M}_{\mathcal{A}})$ to be the coarse map $\alpha \mapsto \mu_\alpha$, where we observe

$\operatorname{diam}(\zeta(\alpha)) \leq 2$ since $\mathcal{A}$ is connected and not a singleton: for β adjacent to α in $\mathcal{A}$, any $\mu, \mu' \in \zeta(\alpha)$ are adjacent to μ_β in $\mathcal{M}_\mathcal{A}$. By our definitions, ζ is coarsely Lipschitz, surjective, and since μ_α is canonical, PMod(Σ)-equivariant; $\mathcal{M}_\mathcal{A}$ is likewise preserved under the action of PMod(Σ) and connected. Let $W \subset \Sigma$ be an essential, non-pants subsurface. For $\alpha = \bigcup_j \alpha_j$ and $\mu = \bigcup_j \mu_j \in \mu_\alpha$ as above, α_j is filling and $\mu_j \setminus \partial F_j$ is complete on F_j, and $\mu_j \cap \partial F_j$ has empty transversals. Hence W intersects μ if and only if it intersects some F_j if and only if it intersects α: the witness set for $\mathcal{M}_\mathcal{A}$ is identical to that of $\mathcal{A}$. Finally, by construction ζ coarsely preserves subsurface projection to annular witnesses; for non-annular witnesses, α and $\mu \in \mu_\alpha$ have uniformly bounded intersection hence coarsely equal projection.

3.2. **Counting intersections.** It remains to show that if μ, μ' are adjacent in $\mathcal{M}_\mathcal{A}$ then $i(\mu, \mu')$ is uniformly bounded, hence $\mathcal{M}_\mathcal{A}$ is admissible, and that there exists a Lipschitz coarse retraction for ζ. If μ, μ' are adjacent then there exists α, α' adjacent in $\mathcal{A}$ such that $\mu \in \mu_\alpha$ and $\mu' \in \mu_{\alpha'}$. Since the components of μ, μ' are parallel to boundaries and the components of α, α' respectively, $i(\operatorname{base}\mu, \operatorname{base}\mu') \leq 4|\mu||\mu'|i(\alpha, \alpha') + 4|\mu||\mu'|(|\alpha| + |\alpha'|) \leq 4\xi(\Sigma)^2(i(\alpha, \alpha') + 2\dim \mathcal{A}(\Sigma))$. Likewise, for $c \in \operatorname{base}\mu$, $d_c(\mu, \mu')$ is uniformly bounded in terms of $i(\alpha, \alpha')$, and likewise for $c' \in \operatorname{base}\mu'$: since $i(\alpha, \alpha')$ is uniformly bounded, so is $i(\mu, \mu')$.

For $\mu \in V(\mathcal{M}_\mathcal{A})$, let $E_\mu = \{\alpha \in V(\mathcal{A}) : \mu \in \mu_\alpha\}$. By an argument identical to that in Section 2.1, $\rho : \mu \mapsto E_\mu$ is a Lipschitz coarse retraction for ζ if E_μ has uniformly bounded diameter. As usual, it suffices to show that there exists $D \geq 0$ such that for any $\mu \in V(\mathcal{M}_\mathcal{A})$ and $\alpha, \alpha' \in E_\mu$, $i(\alpha, \alpha') \leq D$. We verify that for any component $a \in \alpha$ and $a' \in \alpha'$, $i(a, a')$ is bounded uniformly in terms of $\xi(\Sigma)$. If a, a' are both curves, then by construction $a, a' \in \operatorname{base}\mu$ and hence either disjoint or identical. If a is an arc and a' a curve, then a intersects a' at most twice. More generally, for any arc $a \in \alpha$, a is contained in a subsurface F for which $\mu|_F$ is a complete marking on F and $a \setminus \mu$ has at most two components in each component of $F \setminus \mu$. Up to Dehn twists along components in $\operatorname{base}\mu$, a is thus one of finitely many arcs, any two of which have at most 8 intersections in each component of $F \setminus \mu$. The order of each twist is determined up to a uniform constant by the transversals in μ. It follows that any two such arcs have uniformly bounded intersection number, depending only on $\xi(F) \leq \xi(\Sigma)$. Finally, if a, a' are both arcs then choose F to contain both, whence $i(a, a')$ is uniformly bounded. ζ is a quasi-isometry. //

4. Acknowledgements

The author would like to thank Mladen Bestvina for his support and advice, and Kate Vokes, Jacob Russel, and George Shaji for helpful discussions. The author was supported by NSF award nos. 1905720 and 2304774 and no. 1840190: *RTG: Algebra, Geometry, and Topology at the University of Utah.*

References

[BHS17] Jason Behrstock, Mark F. Hagen, and Alessandro Sisto, *Hierarchically hyperbolic spaces I: Curve complexes for cubical groups*, Geometry & Topology **21** (2017), no. 3, 1731–1804.

[BHS19] ______, *Hierarchically hyperbolic spaces II: Combination theorems and the distance formula*, Pacific Journal of Mathematics **299** (2019), no. 2, 257–338.

[BHS21] ______, *Quasiflats in hierarchically hyperbolic spaces*, Duke Mathematical Journal **170** (2021), no. 5, 909 – 996.

[BNV22] Assaf Bar-Natan and Yvon Verberne, *The grand arc graph*, 2022.

[Kop23] Michael C. Kopreski, *Prescribed arc graphs*, 2023.

[MM00] Howard A. Masur and Yair N. Minsky, *Geometry of the complex of curves II: Hierarchical structure*, Geometric & Functional Analysis **10** (2000), 902–974.

[MR20] Kathryn Mann and Kasra Rafi, *Large scale geometry of big mapping class groups*, 2020.

[Sch] Saul Schleimer, *Notes on the complex of curves*, `http://homepages.warwick.ac.uk/~masgar/Maths/notes.pdf`.

[Sis17] Alessandro Sisto, *What is a hierarchically hyperbolic space?*, 2017.

[Vok22] Kate M. Vokes, *Hierarchical hyperbolicity of graphs of multicurves*, Algebraic & Geometric Topology **22** (2022), no. 1, 113–151.